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1983 J. Phys. A: Math. Gen. 16 3601

(http://iopscience.iop.org/0305-4470/16/15/024)

J. Phys. A: Math. Gen. 16 (1983) 3601–3610. Printed in Great Britain

On the critical point of the fully anisotropic quenched bond-random Potts ferromagnet in triangular and honeycomb lattices

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Received 29 March 1983

Abstract. On conjectural grounds we present an equation that provides a very good approximation for the critical temperature of the fully anisotropic homogeneous quenched bond-random q-state Potts ferromagnet in triangular and honeycomb lattices. Almost all the *exact* particular results presently known for the square, triangular and honeycomb lattices are recovered; the numerical discrepancy is quite small for the few exceptions. Some predictions that we believe to be exact are also made explicit.

1. Introduction

A certain amount of effort is presently being devoted to the study of random models, in particular the quenched bond-random q-state Potts model (characterised by the Hamiltonian $\mathscr{H} = -q \sum_{i,j} J_{ij}\delta_{\sigma_i,\sigma_j}$ where $\sigma_i = 1, 2, \ldots, q$ for all sites) in regular lattices (see Southern and Thorpe 1979, Tsallis 1981a, 1983, de Magalhães *et al* 1982 and references therein; for an excellent review see Wu 1982). As the discussion of this class of models is very complex, only a few *exact* facts are known so far. In particular, the exact critical points for the pure model, as well as the limiting critical slopes for the bond-dilute model, have already been established for some two-dimensional lattices (Baxter *et al* 1978, Burkhardt and Southern 1978, Hintermann *et al* 1978, Southern and Thorpe 1979, Tsallis 1982, Wu and Stanley 1982).

In this paper we are concerned with a very general ferromagnetic model in which we associate arbitrary (and independent) probability laws for the coupling constants along the three crystalline axes of the triangular and honeycomb lattices. We focus on the critical temperatures T_c of these two cases. By following along the conjectural lines of Tsallis (1981a) (a quite detailed discussion of the square lattice case) and Tsallis (1983) (a preliminary discussion of the triangular lattice case), we propose relatively simple equations for T_c, which presumably are excellent approximations as they recover a considerable amount of *exact* particular results.

‖ Guggenheim Fellow.

¶ Supported in part by grants from NSF, ONR and ARO.

This paper is organised as follows: in § 2 we introduce a convenient formalism and in §§ 3 and 4 we discuss the triangular and honeycomb cases, respectively.

2. Formalism

In this section we present convenient nomenclature and relations that will be used further on. First let us introduce (Domb 1974) a bond variable, referred to as *thermal transmissivity* (Tsallis and Levy 1980, 1981 and references therein; see also Yeomans and Stinchcombe 1980), through the definition

$$t \equiv [1-\exp(-qJ/k_B T)]/[1+(q-1)\exp(-qJ/k_B T)] \in [0, 1]. \tag{1}$$

If we consider two bonds with coupling constants J_1 and J_2 we obtain, for the equivalent transmissivity t_s of a series array,

$$t_s = t_1 t_2, \tag{2}$$

and, for the transmissivity t_p of a parallel array,

$$t_p = [t_1 + t_2 + (q-2)t_1 t_2]/[1+(q-1)t_1 t_2]. \tag{3}$$

The latter can be rewritten in a series-like form as

$$t_p^D = t_1^D t_2^D \tag{4}$$

where

$$t_i^D \equiv (1-t_i)/[1+(q-1)t_i] \qquad (i = 1, 2, p) \tag{5}$$

and where D stands for 'dual.'

If J is a random variable and $P(J)$ the associated distribution law, then the distribution law for t, denoted $Q(t)$, is given by

$$Q(t) = \frac{k_B T}{(1-t)[1+(q-1)t]} P\left(\frac{k_B T}{q} \ln \frac{1+(q-1)t}{1-t}\right). \tag{6}$$

The corresponding law $Q^D(t^D)$ in the t^D variable is given by

$$Q^D(t^D) = \frac{q}{[1+(q-1)t^D]^2} Q\left(\frac{1-t^D}{1+(q-1)t^D}\right). \tag{7}$$

The distribution law $Q_s(t)$ associated with a series array of two bonds with distribution laws $Q_1(t)$ and $Q_2(t)$ is given by

$$\begin{aligned} Q_s(t) &= \int dt_1 \int dt_2\, Q_1(t_1) Q(t_2) \delta(t - t_1 t_2) \\ &= \int (dt'/t') Q_1(t') Q_2(t/t') = Q_1 \circledS Q_2. \end{aligned} \tag{8}$$

This product (from now on referred to as *series product* or *s product*) recovers, for $q = 1$, that introduced in Tsallis (1981b). Furthermore, it recovers, for $Q_i(t) = \delta(t - t_i)$ $(i = 1, 2)$, equation (2). We can verify that the s product is closed (i.e. it preserves the norm), commutative, associative, admits a neutral element (namely $\delta\ (t-1)$), but not an inverse, i.e. its structure is that of an abelian monoid (semigroup with neutral element); as a matter of fact it is easy to prove (through the transformation $t \equiv e^{-x}$) that it is isomorphic to the convolution product.

If our array is a parallel one the associated law $Q_p(t)$ is given by

$$Q_p(t) = \int dt_1 \int dt_2\, Q_1(t_1) Q_2(t_2) \delta\left(t - \frac{t_1 + t_2 + (q-2)t_1 t_2}{1+(q-1)t_1 t_2}\right)$$
$$= \int dt' \frac{1+(q-2)t' - (q-1)t'^2}{[1+(q-2)t' - (q-1)tt']^2} Q_1(t') Q_2\left(\frac{t-t'}{1+(q-2)t'-(q-1)tt'}\right) \qquad (9)$$
$$\equiv Q_1 ⓟ Q_2.$$

This product (from now on referred to as *parallel product* or *p product*) has the same structure as the s product, the neutral element now being $\delta(t)$; it recovers algorithm (3) and the p product introduced in Tsallis (1981b) as particular cases. It is straightforward to prove that

$$(Q_1 ⓟ Q_2)^D = Q_1^D ⓢ Q_2^D, \qquad (10)$$

thus generalising equation (4) and exhibiting the isomorphism between the s and p products.

It is clear that algorithms (8) and (9) allow the calculation of *any* two-rooted graph (or array) sequentially reducible by series and parallel operations (e.g. that of figure 1).

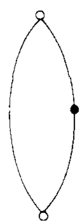

Figure 1. Two-rooted graph. ○ denotes the roots or terminal nodes; ● denotes the internal nodes.

Before closing this section, let us introduce (Tsallis 1981a, Alcaraz and Tsallis 1982, Tsallis and de Magalhães 1981) another convenient variable through

$$s(t) \equiv \ln[1+(q-1)t]/\ln q \in [0, 1]. \qquad (11)$$

It satisfies the following remarkable property

$$s^D(t) \equiv s(t^D) = 1 - s(t), \qquad (12)$$

i.e. *s transforms, under duality, like a probability*; this fact plays an important role in the conjecture we shall present later on. Note also that s coincides with t in the limit $q \to 1$. The distribution law $R(s)$ in the s variable is related to $Q(t)$ through

$$R(s) = [q^s \ln q/(q-1)] Q[(q^s - 1)/(q-1)]. \qquad (13)$$

Furthermore, the distribution law associated with s^D is given by

$$R^D(s^D) = R(1 - s^D). \qquad (14)$$

3. Triangular lattice

Let us consider a triangular lattice to the bonds of which we associate q-state Potts ferromagnetic interactions. The corresponding coupling constants J along the three crystalline axes are respectively and independently distributed according to the laws $P_k(J)$ ($k=1, 2, 3$). Through equations (6) and (13) these laws univocally determine $\{Q_k(t)\}$ and $\{R_k(s)\}$. This quite general model presents a phase transition at a temperature T_c which is still unknown (except for some particular cases described later on). Before stating our proposal for this quantity, let us briefly consider the pure case (i.e. $Q_k(t)=\delta(t-t_k)$). The transmissivities t_Δ and t_{YD} respectively associated with the three-rooted graphs in figure 2(*a*), (*b*) can be calculated by using the break-collapse method (Tsallis and Levy 1981), and are given by

$$t_\Delta(t_1, t_2, t_3)=\frac{t_1t_2+t_2t_3+t_3t_1+(q-3)t_1t_2t_3}{1+(q-1)t_1t_2t_3} \tag{15}$$

$$t_{YD}(t_1^D, t_2^D, t_3^D)=t_1^Dt_2^Dt_3^D. \tag{16}$$

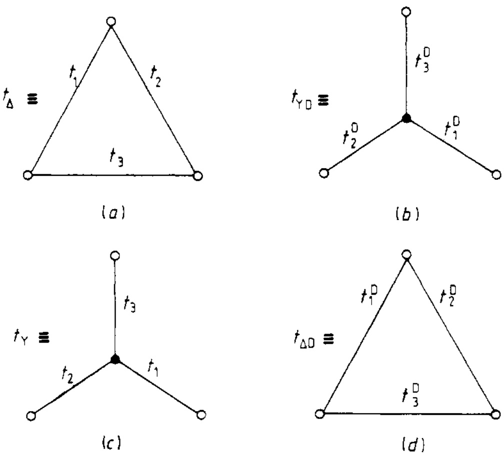

Figure 2. Three-rooted graphs. The t's are the associated transmissivities (D stands for 'dual'; see equation (5)). The pair *a*–*b* (*c*–*d*) is the relevant one for the triangular (honeycomb) lattice.

It is easy to verify that the equation

$$t_\Delta=t_{YD} \tag{17}$$

provides the *exact* critical point (Baxter *et al* 1978, Burkhardt and Southern 1978, Hintermann *et al* 1978). This is essentially a compact way of performing the standard duality and star-triangle transformations.

Let us now go back to the general case where we replace, in figure 2(*a*), $\{t_k\}$ by $\{Q_k(t)\}$ and, in figure 2(*b*), $\{t_k^D\}$ by $\{Q_k^D(t_k^D)\}$. The distributions $Q_\Delta(t)$ and $Q_{YD}(t^D)$ respectively associated with the triangle and star graphs are given by

$$Q_\Delta(t)=\iiint\left(\prod_{k=1}^{3} dt_kQ_k(t_k)\right)\delta[t-t_\Delta(t_1, t_2, t_3)] \tag{18}$$

and

$$Q_{\mathrm{YD}}(t^{\mathrm{D}})=\iiint\left(\prod_{k=1}^{3}\mathrm{d}t_k^{\mathrm{D}}Q_k^{\mathrm{D}}(t_k^{\mathrm{D}})\right)\delta[t^{\mathrm{D}}-t_{\mathrm{YD}}(t_1^{\mathrm{D}},t_2^{\mathrm{D}},t_3^{\mathrm{D}})]. \tag{19}$$

These distributions univocally determine, through use of definitions (1) and (11) and inversion of equations (6) and (13), $P_\Delta(J)$, $P_{\mathrm{YD}}(J^{\mathrm{D}})$, $R_\Delta(s)$ and $R_{\mathrm{YD}}(s^{\mathrm{D}})$ (J^{D} and t^{D} satisfy equation (1)). Note that: (a) Q_{YD} is in general different from $(Q_{\mathrm{Y}})^{\mathrm{D}}$ where Q_{Y} is obtained by associating, with the star graph, $\{Q_k\}$ instead of $\{Q_k^{\mathrm{D}}\}$; (b) $Q_{\mathrm{YD}}=Q_1^{\mathrm{D}} Ⓢ Q_2^{\mathrm{D}} Ⓢ Q_3^{\mathrm{D}}$; (c) the particular case $Q_3(t)=\delta(t)$ (square lattice) leads to $Q_\Delta=Q_1 Ⓢ Q_2$ and $Q_{\mathrm{YD}}=Q_1^{\mathrm{D}} Ⓢ Q_2^{\mathrm{D}}=(Q_1 Ⓟ Q_2)^{\mathrm{D}}$; (d) the particular case $Q_3(t)=\delta(t-1)$ leads to $Q_\Delta=Q_1 Ⓟ Q_2=(Q_1^{\mathrm{D}} Ⓢ Q_2^{\mathrm{D}})^{\mathrm{D}}$ and $Q_{\mathrm{YD}}(t^{\mathrm{D}})=\delta(t^{\mathrm{D}})$.

By conjecturally extending equation (17) we propose, for the critical temperature T_{c} of the general model, the equation

$$\langle s\rangle_\Delta=\langle s\rangle_{\mathrm{YD}} \tag{20}$$

where

$$\begin{aligned}\langle s\rangle_\Delta&\equiv\int_0^1\mathrm{d}s\,sR_\Delta(s)=\int_0^1\mathrm{d}t\{\ln[1+(q-1)t]/\ln q\}Q_\Delta(t)\\&=1-\int_0^\infty\mathrm{d}J\{\ln[1+(q-1)\exp(-qJ/k_{\mathrm{B}}T_{\mathrm{c}})]/\ln q\}P_\Delta(J)\end{aligned} \tag{21}$$

and

$$\begin{aligned}\langle s\rangle_{\mathrm{YD}}&\equiv\int_0^1\mathrm{d}s^{\mathrm{D}}\,s^{\mathrm{D}}R_{\mathrm{YD}}(s^{\mathrm{D}})=\int_0^1\mathrm{d}t^{\mathrm{D}}\{\ln[1+(q-1)t^{\mathrm{D}}]/\ln q\}Q_{\mathrm{YD}}(t^{\mathrm{D}})\\&=1-\int_0^1\mathrm{d}J^{\mathrm{D}}\{\ln[1+(q-1)\exp(-qJ^{\mathrm{D}}/k_{\mathrm{B}}T_{\mathrm{c}})]/\ln q\}P_{\mathrm{YD}}(J^{\mathrm{D}}).\end{aligned} \tag{22}$$

We shall exhibit that equation (20) recovers almost all the *exact* particular results presently known; it fails, however, with respect to the pure Potts limiting slope for the bond-dilute model. The *exact* asymptotic behaviour in the pure percolation limit of the bond-dilute model is recovered, and this is so because, in equation (20), we have averaged the s variable (instead of t, for instance); see Levy *et al* (1980), Tsallis (1981a), de Magalhães *et al* (1982).

Let us first consider the $q\to 1$ limit (hence $s=t$): we verify that equation (20) leads to

$$\sum_{k=1}^{3}\langle t\rangle_{Q_k}-\prod_{k=1}^{3}\langle t\rangle_{Q_k}-1=0 \tag{23}$$

where

$$\langle t\rangle_{Q_k}=\int_0^1\mathrm{d}t_k\,t_kQ_k(t_k)\qquad(k=1,2,3). \tag{24}$$

Consequently, equation (20) satisfies the Kasteleyn and Fortuin (1969) theorem (the $q\to 1$ Potts ferromagnet is isomorphic to bond percolation with $t_k=1-\exp(-J_k/k_{\mathrm{B}}T)$ (see equation (1))) as equation (23) precisely reproduces the form of the bond percolation critical exact equation (Sykes and Essam 1963)

$$\sum_{k=1}^{3}p_k-\prod_{k=1}^{3}p_k-1=0. \tag{25}$$

For the particular case $Q_k(t) = \delta(t - t_k)$ $(k = 1, 2, 3)$, equation (20) clearly leads, for all q, to the exact equation (17). Furthermore, we consider the following generalised bond-dilute model

$$P_k(J) = (1 - p_k)\delta(J) + p_k \bar{P}_k(J) \qquad (k = 1, 2, 3) \tag{26}$$

where the laws $\bar{P}_k(J)$ satisfy, besides the norm condition

$$\int_0^\infty \mathrm{d}J\, \bar{P}_k(J) = 1, \tag{27}$$

the restriction

$$\lim_{\varepsilon \to 0} \int_0^\varepsilon \mathrm{d}J\, \bar{P}_k(J) = 0 \tag{28}$$

(i.e. $\bar{P}_k(J)$ does not grow, in the limit $J \to 0$, as $1/J$ or faster). It is clear that this model must lead, in the limit $T_c \to 0$ and for all q, to the bond percolation equation (25). This is precisely what equation (20) provides, the asymptotic behaviour being

$$\sum_{k=1}^{3} p_k - \prod_{k=1}^{3} p_k - 1 \sim \frac{q-1}{\ln q} \sum_{i \neq j \neq k} (1 - p_i p_j) p_k \int_0^\infty \mathrm{d}J_k\, \bar{P}_k(J_k) \exp(-qJ_k/k_B T_c). \tag{29}$$

For the particular isotropic case $p_k = p$ and $\bar{P}_k(J_k) = \delta(J_k - J), \forall k$, this equation provides

$$\mathrm{d} \exp(-qJ/k_B T_c(p))/\mathrm{d}p|_{p=p_c} = \ln q / p_c(q-1), \tag{30}$$

where p_c denotes the bond percolation critical probability. Equation (30) is known to be *exact* (Southern and Thorpe 1979). As a matter of fact we believe that equation (29) is *exact for the generalised bond-dilute model* (see Tsallis (1981a) for a similar situation in the square lattice). A different situation is found at the opposite limit (maximum T_c, hence $p_k \to 1$, $\forall k$) of the model determined by equation (26). In the case $\bar{P}_k(J) = \delta(J - J_k)$ the limiting T_c is *exact*, but its asymptotic behaviour is *wrong* for all $q \neq 1$, as can be exhibited for the particular isotropic case mentioned above. The error is however very small for $1 < q \leq 4$ (we recall that for $q > 4$ the transition is a first-order one (Baxter 1973)). Equation (20) provides

$$[1/T_c(1)][\mathrm{d}T_c(p)/\mathrm{d}p]|_{p=1}$$
$$= \begin{cases} 1.2472\ldots & (1.2472\ldots; \quad 0\% \text{ error}) \quad \text{for } q = 1 \\ 1.1925\ldots & (1.1877\ldots; \quad 0.40\% \text{ error}) \quad \text{for } q = 2 \\ 1.1634\ldots & (1.1506\ldots; \quad 1.11\% \text{ error}) \quad \text{for } q = 3 \\ 1.1447\ldots & (1.1246\ldots; \quad 1.79\% \text{ error}) \quad \text{for } q = 4 \end{cases} \tag{31}$$

where, between parentheses, we have indicated the exact results (Southern and Thorpe 1979) as well as the discrepancies.

These discrepancies being quite small, we can consider equation (20) to be an approximation for T_c good enough for a great variety of purposes. In particular it leads, for the isotropic bond-dilute model, to

$$3p \ln[1 + (q-1)t] - p^3 \ln[1 + (q-1)t^3] = \ln q, \tag{32}$$

which, for $q = 2$, recovers the renormalisation group result presented in de Magalhães *et al* (1982) (equation (11) therein); let us stress that equation (32) is exact in *both* critical point *and* derivative in the $p \to p_c$ limit but *only* in the critical point in the $p \to 1$ limit.

Before concluding this section, let us mention that equation (20) generalises the Tsallis (1981a) proposal for the square lattice. Indeed, if we consider the particular case $P_3(J) = \delta(J)$ equation (20) can be rewritten as

$$\langle s\rangle_{P_1 \circledS P_2} = \langle s\rangle_{P_2^D \circledS P_1^D} \tag{33}$$

hence

$$\langle s\rangle_{P_1} = \langle s\rangle_{P_2^D} \tag{34}$$

hence

$$\langle s\rangle_{P_1} + \langle s\rangle_{P_2} = 1 \tag{35}$$

which is precisely equation (13) in Tsallis (1981a). It is worthwhile recalling that, with respect to T_c, equation (35) *exactly* satisfies (a) the Kasteleyn and Fortuin (1969) theorem in the limit $q \to 1$, (b) the equal probability model (see Fisch 1978), (c) the bond-dilute model in the $T_c \to 0$ limit (*both* the limit *and* the asymptotic behaviour), (d) the bond-dilute model in the pure Potts limit (*only* the limit; it slightly fails in the asymptotic behaviour for $q \neq 1$).

4. Honeycomb lattice

The honeycomb lattice being the dual of the triangular lattice, this section closely follows the preceding one. Now the laws $P_k(J)$ ($k = 1, 2, 3$) are to be associated with the three crystalline directions of a honeycomb lattice. The transmissivities t_Y and $t_{\Delta D}$ respectively corresponding to figures 2(*c*), (*d*) are given by

$$t_Y(t_1, t_2, t_3) = t_1 t_2 t_3 \tag{36}$$

$$t_{\Delta D}(t_1^D, t_2^D, t_3^D) = \frac{t_1^D t_2^D + t_2^D t_3^D + t_3^D t_1^D + (q-3) t_1^D t_2^D t_3^D}{1 + (q-1) t_1^D t_2^D t_3^D}. \tag{37}$$

It is easy to verify that the pure Potts model ($P_k(J) = \delta(J - J_k)$, $\forall k$) exact critical point (Baxter *et al* 1978, Burkhardt and Southern 1978, Hintermann *et al* 1978) is now provided by the equation

$$t_Y = t_{\Delta D}. \tag{38}$$

For general laws $\{P_k(J)\}$, equations (36) and (37) are respectively extended into

$$Q_Y(t) = \iiint \left(\prod_{k=1}^{3} \mathrm{d}t_k\, Q_k(t_k)\right) \delta[t - t_Y(t_1, t_2, t_3)] \tag{39}$$

$$Q_{\Delta D}(t^D) = \iiint \left(\prod_{k=1}^{3} \mathrm{d}t_k^D\, Q_k^D(t_k^D)\right) \delta[t - t_{\Delta D}(t_1^D, t_2^D, t_3^D)]. \tag{40}$$

Note that (a) $Q_Y = Q_1 \circledS Q_2 \circledS Q_3$; (b) the particular case $Q_3(t) = \delta(t-1)$ (square lattice) leads to $Q_Y = Q_1 \circledS Q_2$ and $Q_{\Delta D} = Q_1^D \circledS Q_2^D$; (c) the particular case $Q_3(t) = \delta(t)$ leads to $Q_Y = \delta(t)$ and $Q_{\Delta D} = Q_1^D \circledP Q_2^D = (Q_1 \circledS Q_2)^D$.

The proposal for T_c will now be

$$\langle s\rangle_Y = \langle s\rangle_{\Delta D} \tag{41}$$

where

$$\langle s\rangle_{\mathrm{Y}} \equiv \int_0^1 \mathrm{d}s\, sR_{\mathrm{Y}}(s) = \int_0^1 \mathrm{d}t\, \{\ln[1+(q-1)t]/\ln q\}Q_{\mathrm{Y}}(t)$$
$$= 1 - \int_0^\infty \mathrm{d}J\, \{\ln[1+(q-1)\exp(-qJ/k_{\mathrm{B}}T_{\mathrm{c}})]/\ln q\}P_{\mathrm{Y}}(J) \tag{42}$$

and

$$\langle s\rangle_{\Delta\mathrm{D}} \equiv \int_0^1 \mathrm{d}s^{\mathrm{D}}\, s^{\mathrm{D}}R_{\Delta\mathrm{D}}(s^{\mathrm{D}}) = \int_0^1 \mathrm{d}t^{\mathrm{D}}\, \{\ln[1+(q-1)t^{\mathrm{D}}]/\ln q\}Q_{\Delta\mathrm{D}}(t^{\mathrm{D}})$$
$$= 1 - \int_0^\infty \mathrm{d}J^{\mathrm{D}}\, \{\ln[1+(q-1)\exp(-qJ^{\mathrm{D}}/k_{\mathrm{B}}T_{\mathrm{c}})]/\ln q\}P_{\Delta\mathrm{D}}(J^{\mathrm{D}}) \tag{43}$$

(the definitions of the quantities R_{Y}, P_{Y}, $R_{\Delta\mathrm{D}}$ and $P_{\Delta\mathrm{D}}$ are self explanatory within the adopted notation).

As for the triangular lattice case, equation (41) recovers almost all the *exact* particular results presently known; it fails however with respect to the pure Potts limiting slope for the bond-dilute model. The $q \to 1$ limit provides

$$\sum_{i<j} \langle t\rangle_{Q_i}\langle t\rangle_{Q_j} - \prod_{k=1}^{3} \langle t\rangle_{Q_k} - 1 = 0, \tag{44}$$

which precisely reproduces the form of the bond percolation critical exact equation (Sykes and Essam 1963),

$$\sum_{i<j} p_i p_j - \prod_{k=1}^{3} p_k - 1 = 0, \tag{45}$$

and therefore the Kasteleyn and Fortuin (1969) theorem is satisfied.

If we consider the model characterised by equation (26), equation (41) leads, in the $T_{\mathrm{c}} \to 0$ limit, to

$$\sum_{i<j} p_i p_j - \prod_{k=1}^{3} p_k - 1 \sim [(q-1)/\ln q] \sum_{i\neq j\neq k} (p_i + p_j - p_i p_j)p_k \int_0^\infty \mathrm{d}J_k\, \bar{P}_k(J_k)\exp(-qJ_k/k_{\mathrm{B}}T_{\mathrm{c}}). \tag{46}$$

For the particular isotropic case $p_k = p$ and $\bar{P}_k(J_k) = \delta(J_k - J)$ $(\forall k)$, this equation provides the exact result (Southern and Thorpe 1979), namely equation (30), p_{c} now being the critical probability corresponding to the honeycomb lattice. As for the triangular lattice case, we believe that equation (46) is *exact for the generalised bond-dilute model.* This is not so for the opposite limit (maximum T_{c} hence $p_k \to 1$, $\forall k$) of this model. In particular for the case $\bar{P}_k(J) = \delta(J - J_k)$ the limiting T_{c} is *exact*, but not the asymptotic behaviour for $q \neq 1$. For the particular isotropic case we obtain, from equation (41),

$$[1/T_{\mathrm{c}}(1)][\mathrm{d}T_{\mathrm{c}}(p)/\mathrm{d}p]|_{p=1}$$
$$= \begin{cases} 1.7770\ldots & (1.7770\ldots;\quad 0\%\ \text{error}) & \text{for } q=1 \\ 1.5998\ldots & (1.5782\ldots;\quad 1.37\%\ \text{error}) & \text{for } q=2 \\ 1.5142\ldots & (1.4659\ldots;\quad 3.30\%\ \text{error}) & \text{for } q=3 \\ 1.4609\ldots & (1.3863\ldots;\quad 5.39\%\ \text{error}) & \text{for } q=4 \end{cases} \tag{47}$$

where, between parentheses, we have indicated the exact results (Southern and Thorpe 1979) as well as the discrepancies. It is straightforward to obtain, from equation (41), the whole critical line:

$$3p^2(1-p)\ln[1+(q-1)t^2]+p^3\ln[1+3(q-1)t^2+(q-1)(q-2)t^3]=\ln q. \tag{48}$$

This equation recovers, for $q=2$, the renormalisation group result presented in de Magalhães *et al* (1982, equation (14)); let us stress that equation (48) is exact in *both* critical point *and* derivative in the $p\to p_c$ limit, but *only* in the critical point in the $p\to 1$ limit.

The square lattice result (equation (35)) can be reobtained by taking $Q_3(t)=\delta(t-1)$ in equation (41).

5. Conclusion

The fully anisotropic homogeneous quenched bond-random q-state Potts ferromagnet is a fairly general model, and its critical temperature T_c is unknown for all lattices with dimensionality higher than one. However, a certain amount of particular exact results are already available for some lattices such as the triangular and honeycomb ones. Following along the conjectural lines of Tsallis (1981a) we propose equations for T_c (equation (20) for the triangular lattice and equation (41) for the honeycomb; both equations contain the Tsallis (1981a) proposal for the square lattice as a particular case) which are believed to provide numerically excellent approximations (at least for $1<q\leqslant 4$; they are exact for $q=1$). They both satisfy the Kasteleyn and Fortuin (1969) theorem, which is herein expressed in a quite general form (the $q\to 1$ limit of the quenched bond-random Potts ferromagnet is isomorphic to bond percolation). They both recover the *exact* T_c for the anisotropic (arbitrary non-negative J_1, J_2 and J_3) pure Potts model and the *exact* percolation critical surface (in the p_1–p_2–p_3 space) in the $T_c\to 0$ limit of a generalised bond-dilute model (characterised by equation (26)). Futhermore, they provide new particular asymptotic behaviours (equation (29) for the triangular lattice, and equation (46) for the honeycomb one), which are possibly exact. Finally, for the standard isotropic bond-dilute model, they provide simple analytical equations (equation (32) for the triangular lattice and equation (48) for the honeycomb one), which although not exact (in the $p\to p_c$ limit both the critical point and asymptotic behaviour are exact, but in the $p\to 1$ limit only the critical point is exact, the corresponding asymptotic behaviour presenting a numerically small failure), can be useful for several purposes as long as the exact equations remain unknown; the biggest estimated error (in the t variable) they introduce presumably occurs midway between $p=p_c$ and $p=1$ and increases from 0% for $q=1$ to about 1% for the triangular lattice (0.5% for the honeycomb lattice) for $q=4$.

Acknowledgments

It is with pleasure that we acknowledge the hospitality of Professors H E Stanley, W Klein, S Redner and other members of the Center for Polymer Studies at Boston University during our stay there. One of us (CT) also acknowledges related and useful conversations with F Y Wu and I Syozi. RJVS gratefully acknowledges the partial support he has received from CAPES/Brazil and OEA.

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